Classification

Introduction to Supervised Learning

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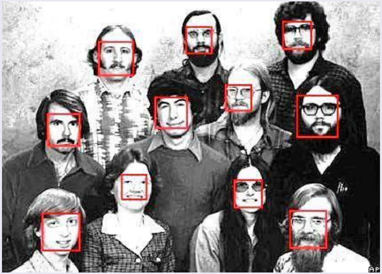
Credit Default, Credit Score, Bank Risk, Market Risk Management

- Data: Client profile, Client credit history...
- Input: Client profile
- Output: Credit risk

Spam detection (Text classification)

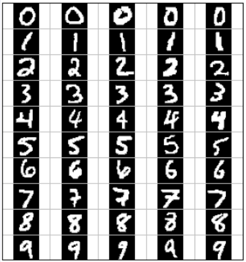
- Data: email collection
- Input: email
- Output : Spam or No Spam

Face Detection



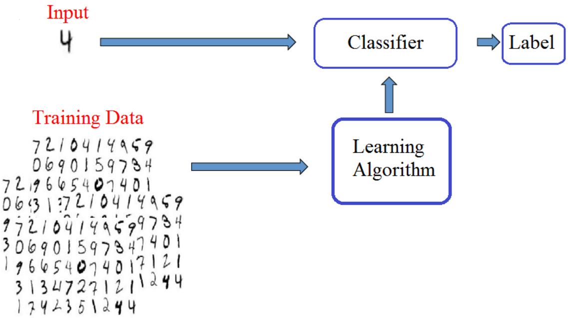
- Data: Annotated database of images
- Input : Sub window in the image
- Output : Presence or no of a face...

Number Recognition



- Data: Annotated database of images (each image is represented by a vector of $28 \times 28 = 784$ pixel intensities)
- Input: Image
- Output: Corresponding number

Machine Learning



A definition by Tom Mitchell (`http://www.cs.cmu.edu/~tom/`)

A computer program is said to learn from experience E with respect to some class of tasks T and performance measure P, if its performance at tasks in T, as measured by P, improves with experience E.

Supervised Learning Framework

- Input measurement $\mathbf{X} = (X^{(1)}, X^{(2)}, \ldots, X^{(d)}) \in \mathcal{X}$
- Output measurement $Y \in \mathcal{Y}$.
- $(\mathbf{X}, Y) \sim \mathbf{P}$ with $\mathbf{P}$ unknown.
- Training data : $\mathcal{D}_n = \{(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)\}$ (i.i.d. $\sim \mathbf{P}$)
- Often
  - $\mathbf{X} \in \mathbb{R}^d$ and $Y \in \{-1, 1\}$ (classification)
  - or $\mathbf{X} \in \mathbb{R}^d$ and $Y \in \mathbb{R}$ (regression).

- A classifier is a function in $\mathcal{F} = \{f : \mathcal{X} \to \mathcal{Y}\}$

Goal

- Construct a good classifier $\widehat{f}$ from the training data.

- Need to specify the meaning of good.
- Formally, classification and regression are the same problem!

Loss function

- Loss function : $\ell(f(x), y)$ measure how well $f(x)$ “predicts" y.
- Examples:
  - Prediction loss: $\ell(Y, f(\mathbf{X})) = \mathbf{1}_{Y \neq f(\mathbf{X})}$
  - Quadratic loss: $\ell(Y, \mathbf{X}) = |Y - f(\mathbf{X})|^2$

Risk of a generic classifier

- Risk measured as the average loss for a new couple:

$$\mathcal{R}(f) = \mathbb{E}\left[\ell(Y, f(\mathbf{X}))\right]$$

- Examples:
  - Prediction loss: $\mathbb{E}\left[\ell(Y, f(\mathbf{X}))\right] = \mathbb{P}\left\{Y \neq f(\mathbf{X})\right\}$
  - Quadratic loss: $\mathbb{E}\left[\ell(Y, f(\mathbf{X}))\right] = \mathbb{E}\left[|Y - f(\mathbf{X})|^2\right]$

- **Beware:** As $\widehat{f}$ depends on $\mathcal{D}_n$

Experience, Task and Performance measure

- Training data : $\mathcal{D} = \{(\mathbf{X}_1, Y_1), \ldots, (\mathbf{X}_n, Y_n)\}$ (i.i.d. $\sim \mathbf{P}$)
- Predictor: $f : \mathcal{X} \to \mathcal{Y}$
- Cost/Loss function : $\ell(f(\mathbf{X}), Y)$
- Risk: $\mathcal{R}(f) = \mathbb{E}\left[\ell(Y, f(\mathbf{X}))\right]$

- Often $\ell(f(\mathbf{X}), Y) = |f(\mathbf{X}) - Y|^2$ or $\ell(f(\mathbf{X}), Y) = \mathbf{1}_{Y \neq f(\mathbf{X})}$

Goal

- Learn a rule to construct a classifier $\widehat{f} \in \mathcal{F}$ from the training data $\mathcal{D}_n$ s.t. the risk $\mathcal{R}(\widehat{f})$ is small on average or with high probability with respect to $\mathcal{D}_n$.

Machine Learning

- Learn a rule to construct a classifier $\widehat{f} \in \mathcal{F}$ from the training data $\mathcal{D}_n$ s.t. the risk $\mathcal{R}(\widehat{f})$ is small on average or with high probability with respect to $\mathcal{D}_n$.

Canonical example: Empirical Risk Minimizer

- One restricts f to a subset of functions $\mathcal{S} = \{f_\theta, \theta \in \Theta\}$
- One replaces the minimization of the average loss by the minimization of the empirical loss

$$\widehat{f} = f_{\widehat{\theta}} = \underset{f_\theta, \theta \in \Theta}{\text{argmin}} \frac{1}{n} \sum_{i=1}^{n} \ell(Y_i, f_\theta(\mathbf{X}_i))$$

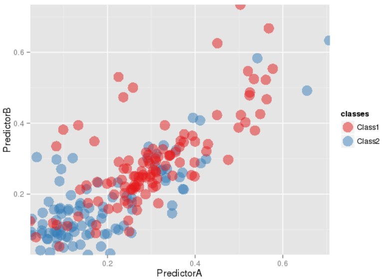
- Examples:
  - Linear discrimination with

$$\mathcal{S} = \{\mathbf{x} \mapsto \mathtt{sign}\{\beta^T \mathbf{x} + \beta_0\} \,/ \beta \in \mathbb{R}^d, \beta_0 \in \mathbb{R}\}$$

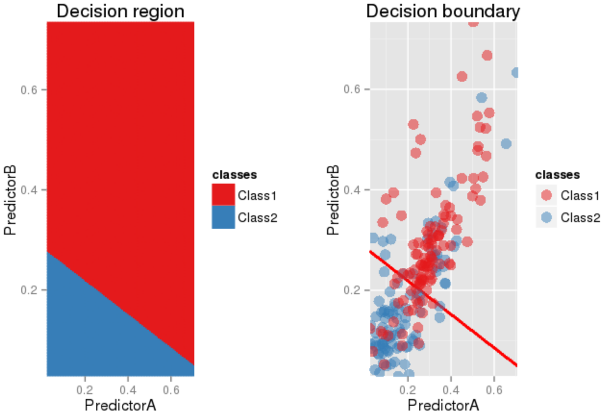
Synthetic Dataset

- Two features/covariates.
- Two classes.

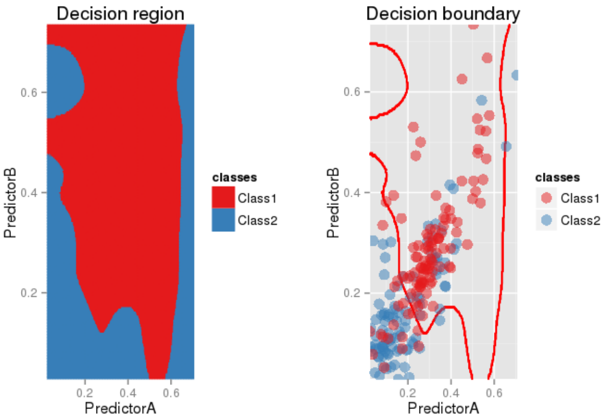
- Dataset from *Applied Predictive Modeling*, M. Kuhn and K. Johnson, Springer
- Numerical experiments with **R** and the **caret** package.



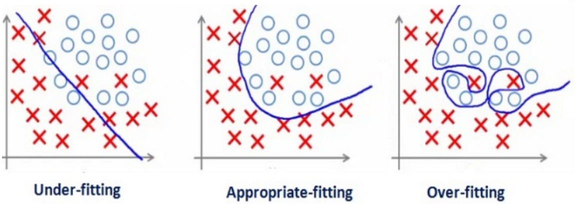
Example: Linear Discrimination



Example: More Complex Model

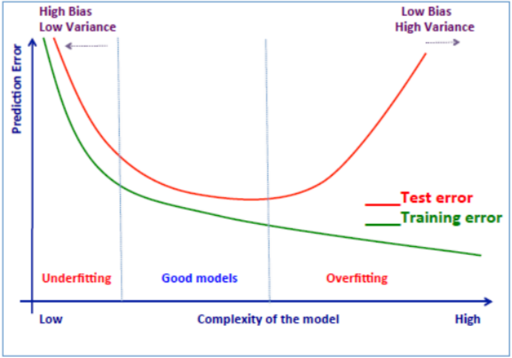


Under-fitting / Over-fitting Issue



- Different behavior for different model complexity
- Under-fit : Low complexity models are easily learned but too simple to explain the truth.
- Over-fit : High complexity models are memorizing the data they have seen and are unable to generalize to unseen examples.

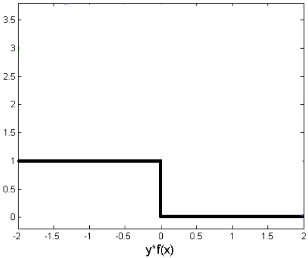
Under-fitting / Over-fitting Issue



- We can determine whether a predictive model is underfitting or overfitting the training data by looking at the prediction error on the training data and the test data.
- How to estimate the test error ?

Empirical Risk Minimizer

$$\widehat{f} = \underset{f \in \mathcal{S}}{\operatorname{argmin}} \frac{1}{n} \sum_{i=1}^{n} \ell^{0/1}(Y_i, f(\mathbf{X}_i))$$



- Classification loss: $\ell^{0/1}(y, f(x)) = \mathbf{1}_{y \neq f(x)}$
- Not convex and not smooth!

- The best solution f^* (which is independent of $\mathcal{D}_n$) is

$$f^* = \arg\min_{f \in \mathcal{F}} R(f) = \arg\min_{f \in \mathcal{F}} \mathbb{E}\left[\ell(Y, f(\mathbf{X}))\right]$$

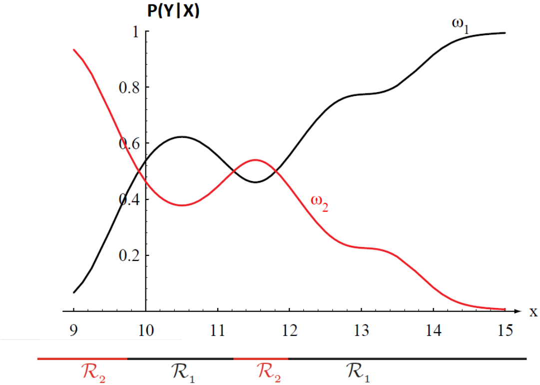
Bayes Predictor (Ideal solution)

In binary classification with $0-1$ loss:

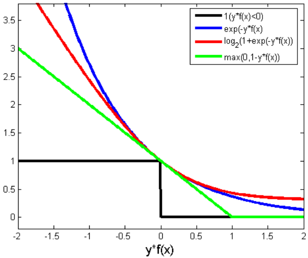
$$f^*(\mathbf{X}) = \begin{cases} +1 & \text{if } \mathbb{P}\left\{Y = +1 | \mathbf{X}\right\} \geq \mathbb{P}\left\{Y = -1 | \mathbf{X}\right\} \\ -1 & \text{otherwise} \end{cases}$$

Issue: Explicit solution requires to know $\mathbb{E}\left[Y | \mathbf{X}\right]$ for all values of $\mathbf{X}$!

Conditional prob. and Bayes Predictor



Classification Loss and Convexification

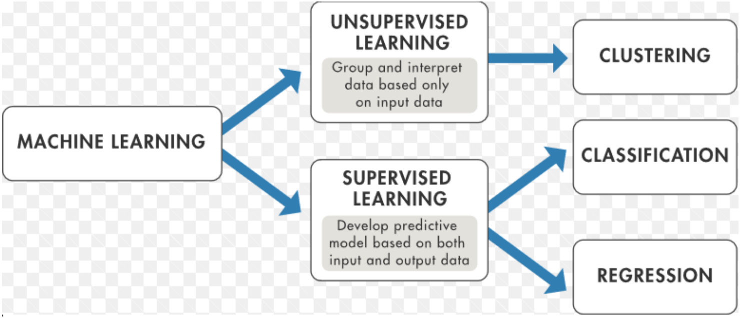


- Classification loss: $\ell^{0/1}(y, f(x)) = \mathbf{1}_{y \neq f(x)}$
- Not convex and not smooth!

Classical convexification

- Logistic loss: $\ell(y, f(x)) = \log(1 + e^{-yf(x)})$ (Logistic / NN)
- Hinge loss: $\ell(y, f(x)) = (1 - yf(x))_+$ (SVM)
- Exponential loss: $\ell(y, f(x)) = e^{-yf(x)}$ (Boosting...)

Machine Learning



Methods (Today):

1. k Nearest-Neighbors

Bibliography

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